Putting Order into Classifications and Universality

AUTOMATA 2010

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LAMA lab. (CNRS, Université de Savoie, France)

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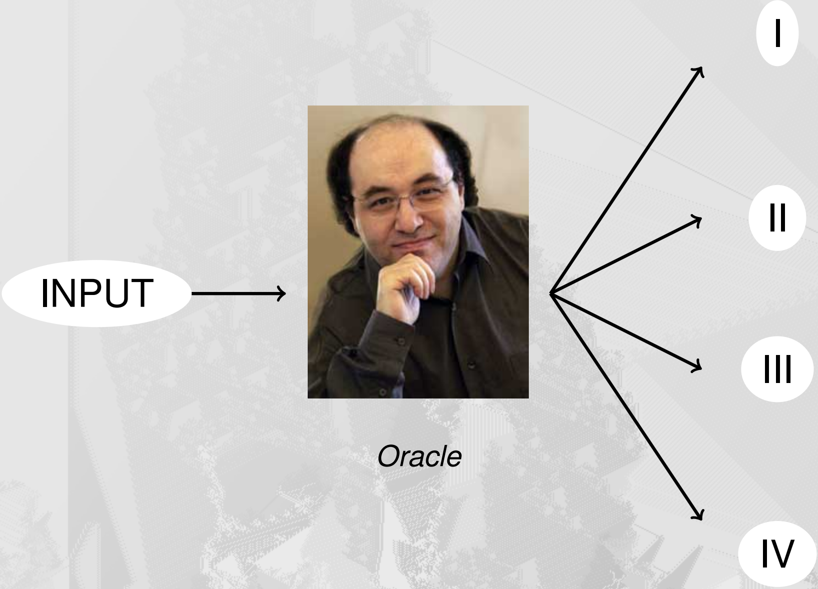
Chronology

- Banks (1970)
- Albert and Čulik (1987)
- Martin I (1993)
- Durand and Róka (1996)
- Durand-Lose (1997)
- Mazoyer and Rapaport (1998)
- Ollinger (2002)
- T (2005)

What is a classification?



What is a classification?

- **Classical approach:**
  1. define *a priori* a list of properties or classes
  2. being similar $\Leftrightarrow$ satisfying the same properties

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  2. equivalence classes are obtained *a posteriori*

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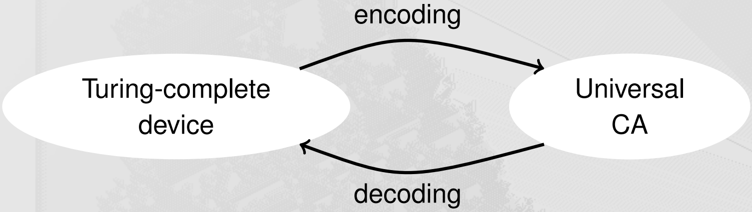
- **This talk:**
  1. define a relation of similarity or equivalence between CA
  2. equivalence classes are obtained *a posteriori*

Key points:

- no external tools used to define properties
- infinitely many classes

What is universality?

- **Classical approach:**



What is universality?

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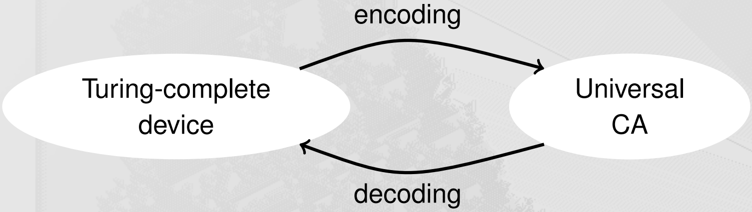

- encoding/decoding/halt problem
- only a positive definition

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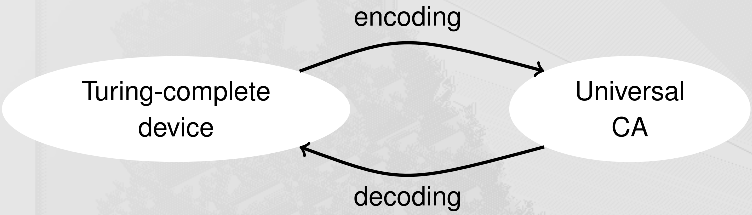

- encoding/decoding/halt problem
- only a positive definition

▶ **This talk:**

1. define a notion of **simulation** between CA
2. **intrinsic universality** $\overset{def}{=}$ ability to simulate *all* CA

Putting Order into Classifications and Universality

- *1 solution for 2 problems*

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     $$F \sim G \overset{def}{\Leftrightarrow} F \preceq G \text{ and } G \preceq F$$

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   - topology of the pre-order

3. universality:

   $$F \text{ universal } \overset{def}{\Leftrightarrow} \forall G, G \preceq F$$

Simulation pre-orders

- *Ingredients:*

1. "local" comparison relations
   - subautomaton
   - factor
2. rescaling operations

General idea

pre-order $\equiv$ local comparison up to rescaling

Subautomaton

- $F \sqsubseteq G$

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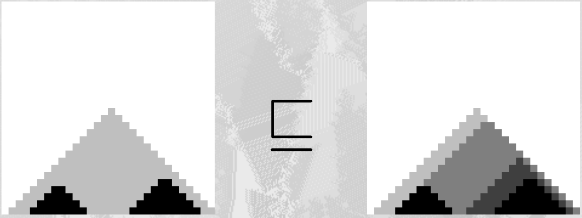
F G

$$\text{injection } \iota : Q_F \to Q_G \text{ with } \iota \circ F = G \circ \iota$$

- example:



MAX with 3 states $\sqsubseteq$ MAX with 5 states

Factor

- $F \trianglelefteq G$

Factor

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surjection $\pi : Q_G \to Q_F$ *with* $\pi \circ G = F \circ \pi$

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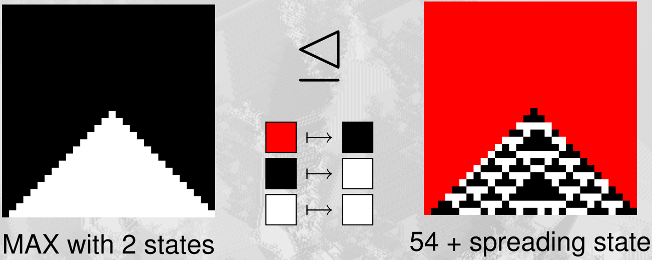
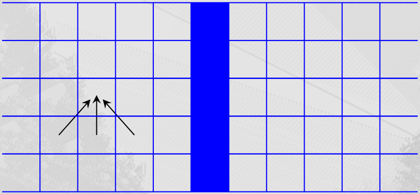


Rescaling operations

- **3 parameters:** $F \mapsto F^{\langle m,t,z \rangle}$



Global map

$$F^{\langle 1,1,0 \rangle} = F$$

Rescaling operations

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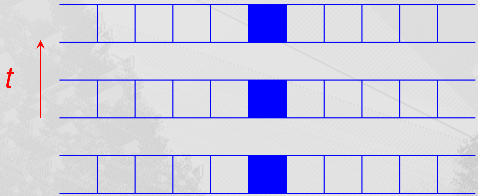
- time



Global map

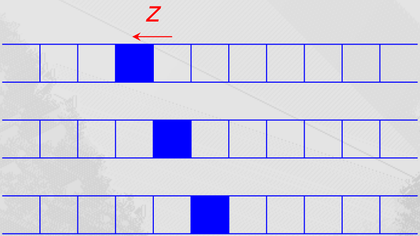
$$F^{\langle 1,t,0 \rangle} = F^{t}$$

Rescaling operations

- **3 parameters:** $F \mapsto F^{\langle m,t,z \rangle}$

■ time

■ shift



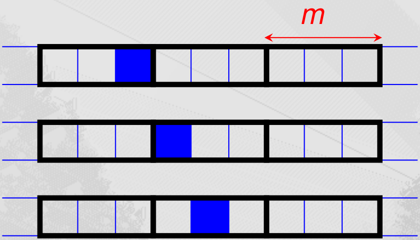
Global map

$$F^{\langle 1,t,z \rangle} = \sigma_z \circ F^t$$

Rescaling operations

▶ **3 parameters:** $F \mapsto F^{\langle m,t,z \rangle}$

- time
- shift
- cell grouping



Global map

$$F^{\langle m,t,z \rangle} = \mathbf{o}_m^{-1} \circ \sigma_z \circ F^t \circ \mathbf{o}_m$$

$\mathbf{o}_m^{-1} : Q^{\mathbb{Z}} \rightarrow (Q^m)^{\mathbb{Z}}$ canonical bijection

Rescaling operations

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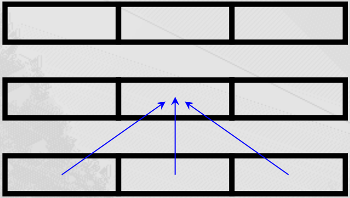


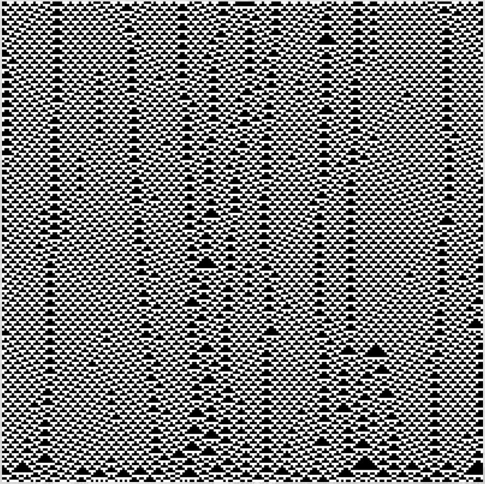
Fact

$F^{\langle m,t,z \rangle}$ is a cellular automaton

- with a possibly different alphabet
- with a possibly different radius

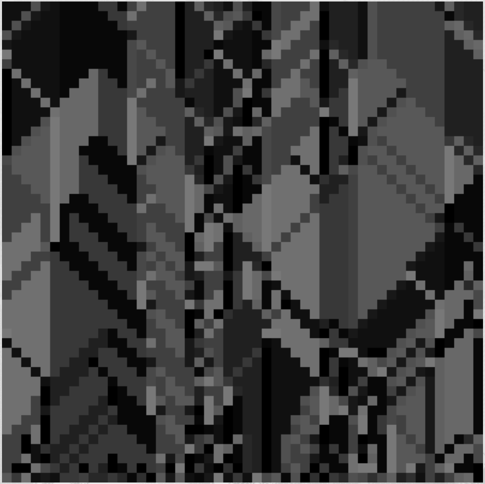
Rescaling examples

54

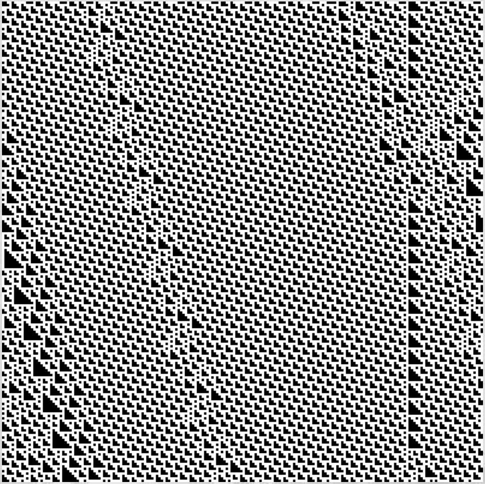


Rescaling examples

$54^{\langle 4,4,0\rangle}$

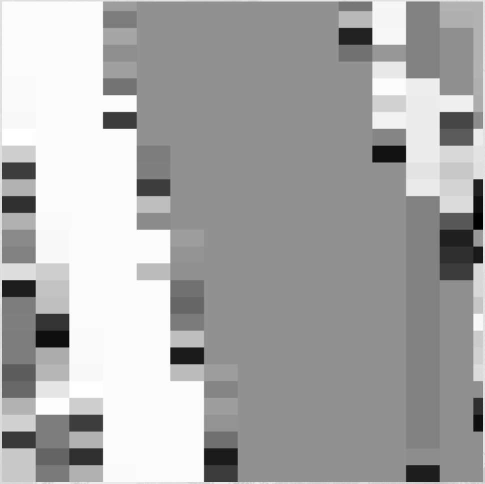


110



Rescaling examples

$110^{\langle 14,7,0 \rangle}$



3 pre-orders

- **injective** simulation

$$F \preceq_i G \overset{def}{\Leftrightarrow} \exists \overrightarrow{p_1}, \overrightarrow{p_2} : F^{\langle \overrightarrow{p_1} \rangle} \sqsubseteq G^{\langle \overrightarrow{p_2} \rangle}$$

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- **surjective** simulation

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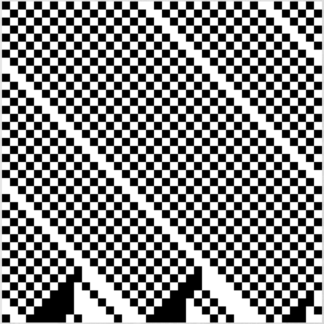
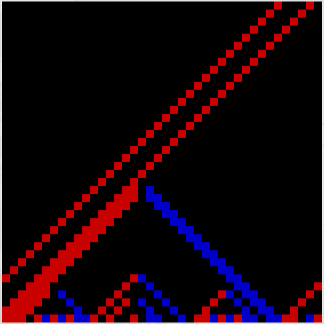
- **mixed** simulation

$$F \preceq_m G \overset{def}{\Leftrightarrow} \exists \overrightarrow{p_1}, \overrightarrow{p_2} : F^{\langle \overrightarrow{p_1} \rangle} \trianglelefteq\sqsubseteq G^{\langle \overrightarrow{p_2} \rangle}$$

Examples

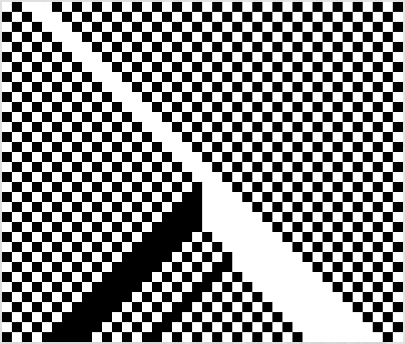


ECA 184

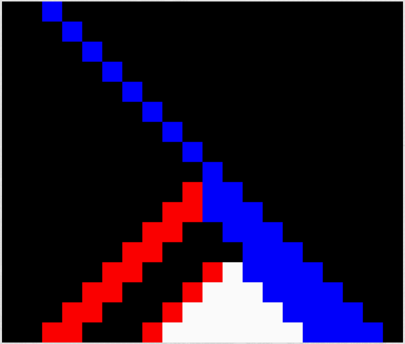


’Just Gliders’

Examples



'Just Gliders' $\trianglelefteq$ $184^{\langle 2,2\rangle}$ $\sim$ 184



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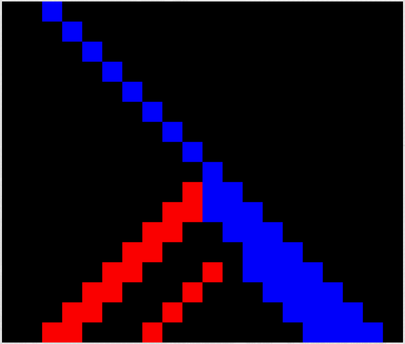

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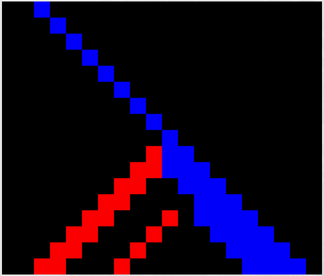
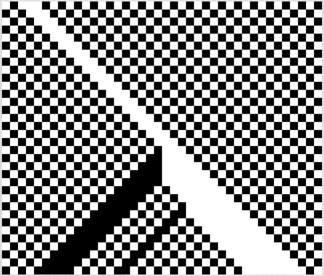
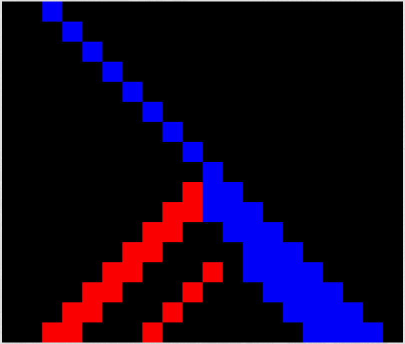
■ = □



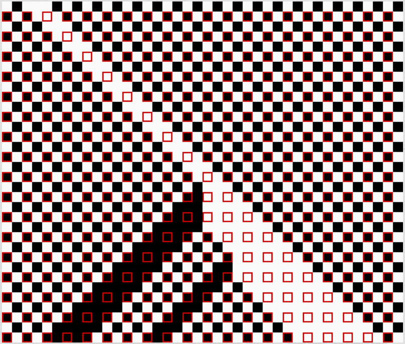


'Just Gliders' $\preceq_s$ 184



'Just Gliders' $\sqsubseteq$ $184^{\langle 2,2 \rangle}$ $\sim$ 184

Examples



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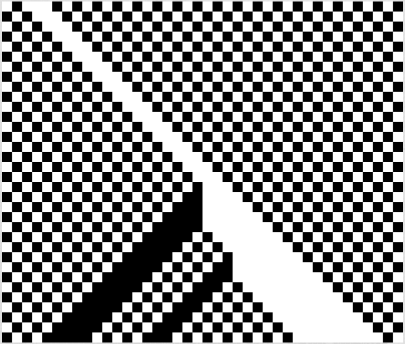

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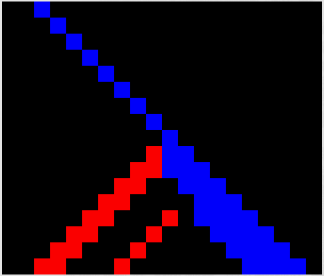
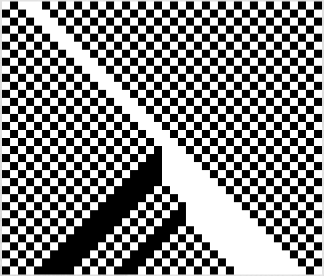
Examples

'Just Gliders' $\preceq_i$ 184

Separation

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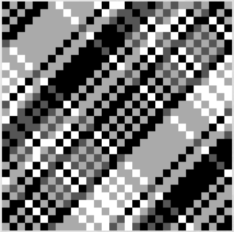
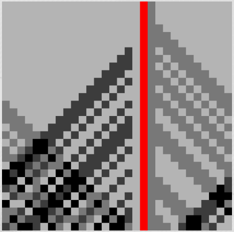


$\sigma \times \sigma^{-1}$

$\preceq_i$

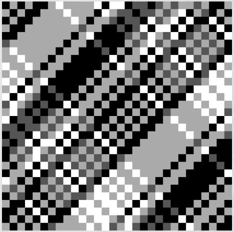
$\not\preceq_s$

$\preceq_m$



$\sigma \times \sigma^{-1}$ + wall state

Separation



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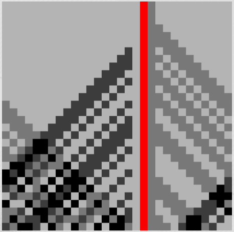

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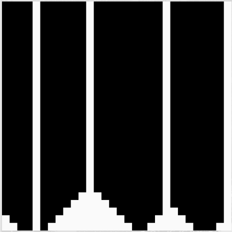
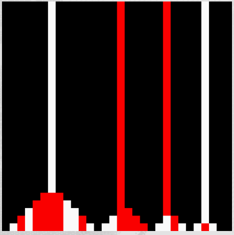


block reduction

$\not\preceq_i$

$\preceq_s$

$\preceq_m$



block reduction + parity test

Pre-order $\preceq_i$

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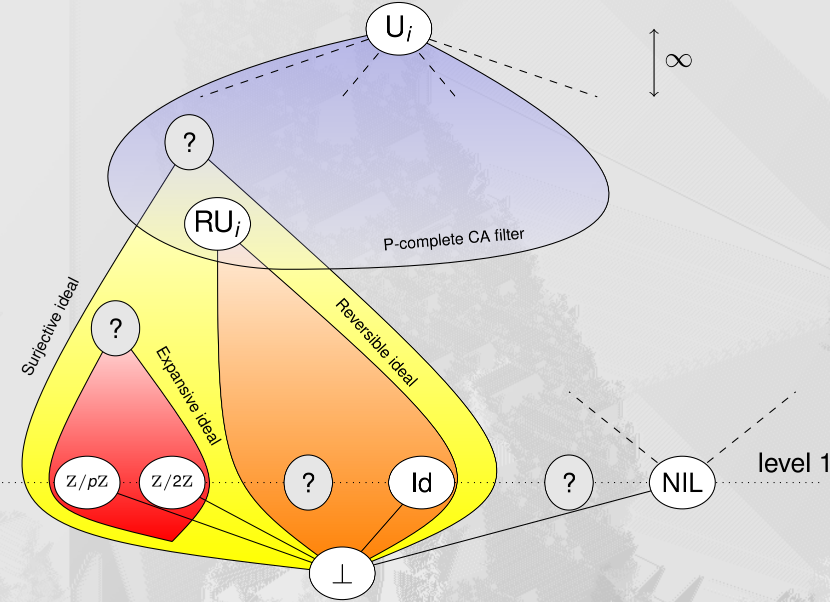

Pre-order $\preceq_m$

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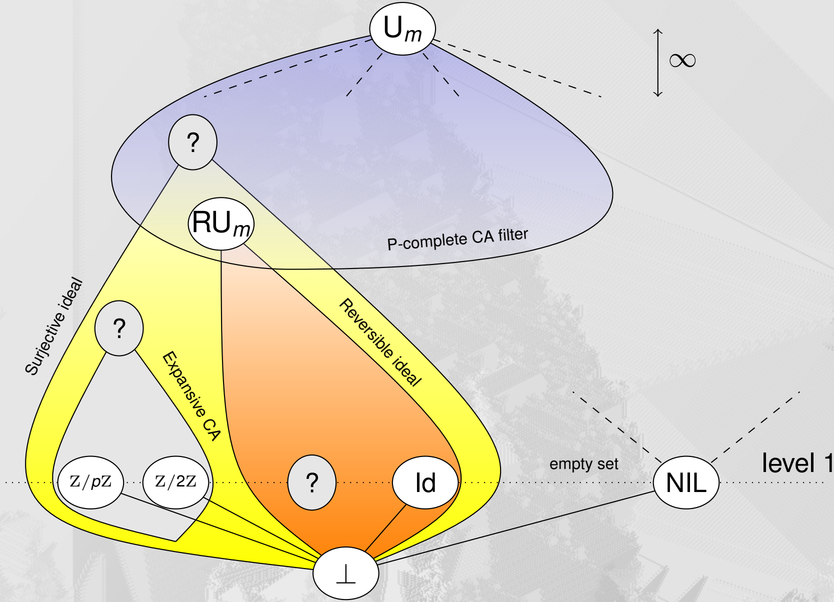

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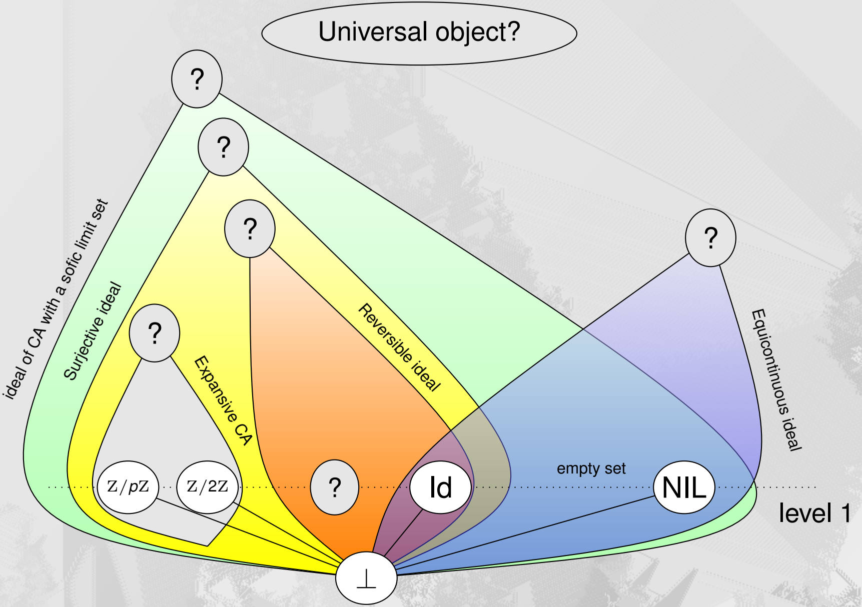

Intrinsic Universality

intrinsic universality $\stackrel{def}{=}$ U_i $\stackrel{def}{=}$ $\left\{F : \forall G, G \preceq_i F\right\}$

N. Ollinger, “Universalities in Cellular Automata”,
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Theorem (Rapaport) — no real-time universality

If $F \in U_i$ then there is some G with $G^{\langle \overrightarrow{p} \rangle} \not\trianglelefteq F^{\langle t,t,z \rangle}$, $(\forall \overrightarrow{p}, t, z)$.

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Theorem (Ollinger) — strong universality

If $F \in U_i$ then for all G there is $\overrightarrow{p}$ such that $G \sqsubseteq F^{\langle \overrightarrow{p} \rangle}$.

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Theorem (Ollinger) — strong universality

If $F \in U_i$ then for all G there is $\overrightarrow{p}$ such that $G \sqsubseteq F^{\langle \overrightarrow{p} \rangle}$.

Exercise

If $F \in U_i$ and has a spreading state then it can simulate any G without using the spreading state.

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Intrinsic universality is undecidable.

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Proposition

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Intrinsic Universality

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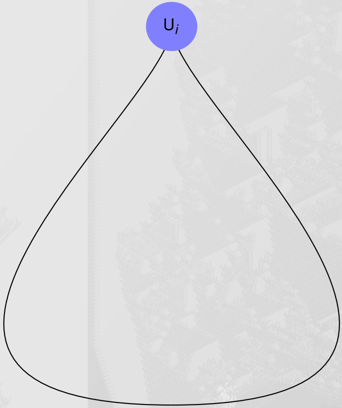
Exercise

If F is not universal then there is a non-universal G with

$$F \preceq_i G \text{ but } G \not\preceq_i F$$

How Common is Intrinsic Universality?



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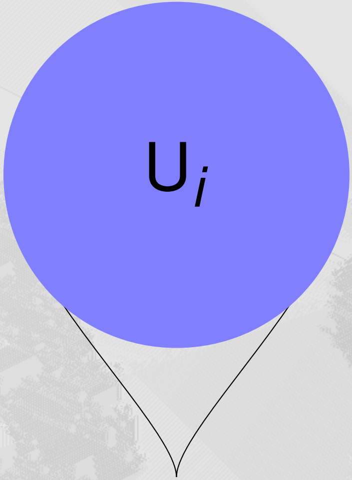

How Common is Intrinsic Universality?

The quest for small intrinsically universal CA

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  - Conway 1970: Game of Life (2 states + Moore neighb.)

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  - Banks 1970: 2 states + large radius
  - Ollinger 2002: 6 states + radius 1
  - Richard 2008: 4 states + radius 1 (U_i or U_m?)

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Open problems

- $U_i \stackrel{?}{=} U_m$
- is 110 intrinsically unievrsal?

How Common is Intrinsic Universality?

Adding syntactical constraints

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Adding syntactical constraints

- **Definitions:**
  - F is **captive** if $\forall \overrightarrow{x}$:

$$f(x_1, \ldots, x_k) \in \{x_1, \ldots, x_k\}$$

  - F is **multiset** if $\forall \overrightarrow{x}$ and $\forall$ permutation π

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Theorem

There exists intrinsically universal CA in the following families:

- number conserving
- totalistic ($\subseteq$ multiset)
- captive

How Common is Intrinsic Universality?

Density

- $\mathcal{P}$ a **property** (a set of CA)
- $\mathcal{F}$ a **family** (a set of CA)
- $\mathcal{F}_{n,r}$: CA from $\mathcal{F}$ with n states and radius r

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$$d_n(\mathcal{P}/\mathcal{F}) = \lim_{n\to\infty} \frac{\#\mathcal{P} \cap \mathcal{F}_{n,r_0}}{\#\mathcal{F}_{n,r_0}}$$

$$d_r(\mathcal{P}/\mathcal{F}) = \lim_{r\to\infty} \frac{\#\mathcal{P} \cap \mathcal{F}_{n_0,r}}{\#\mathcal{F}_{n_0,r}}$$

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Theorem (Boyer,T.)

- $d_n(U_i/\text{captive CA}) = 1$
- $d_r(U_i/\text{multiset CA}) = 1$
- combination of both + other families (e.g. outer-totalistic)

How Common is Intrinsic Universality?

Density

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Open problem

- $d_n(U_i/CA)$?
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Theorem

There are recursive fair encodings from CA into captive CA, and from CA into multiset CA.

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What are equivalence classes without any captive and/or multiset CA?

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Open problem

Is there a "surjective-universal" CA in dimension 1?

Other universalities

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  - q **persistent state** $\overset{def}{\Leftrightarrow} f(*, q, *) = q$
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Is there a universal CA for $\preceq_s$?

Future directions

1. more general simulations?
   - sub-systems induced by stable sub-shifts (SFT? sofic?)
   - factor with context (sliding block codes)
2. dimension change
   - constructions à la Toffoli?
   - sub-actions à la Hochman?
3. asynchronous CA

Don’t Know What to Do During the World Cup?

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- **My list of small open questions**

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Thank you!